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A Note on Prime Graph of a Finite Ring

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Abstract

Sathyanaryana et al. (2010) introduced the concept of prime graph of an associative ring R and studied some properties in their paper. We found that a correction to be made in the statement of the first theorem of their paper. In this paper, we investigate some properties of prime graphs of finite rings and give a corrected version of the said result. We establish a formula for finding the number of edges in the prime graph of the ring of residue classes modulo n by using the gcd-sum function. Further, we discuss some results regarding eigenvalues and energy of prime graphs of finite rings.

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1 Introduction

For standard terminology and notion in groups, rings, graphs and matrices, we refer the reader to the text-books of Herstein [10], Lambek [13], Harary [9] and Bapat [2]. The non-standard will be given in this paper as and when required.

Throughout this paper, R denotes a non-zero finite associative ring. We denote the additive zero of R by 0. The order of an element a in a group Γ is denoted by $o(a)$. The order of a set S is denoted by $|S|$. The greatest common divisor (gcd) of two numbers x and y is denoted by $\gcd(x, y)$. The Eulers phi-function is denoted by ϕ. For a positive integer n, the ring of residue classes modulo n is denoted by $\mathbb{Z}_n$. For a graph G, $V(G)$ and $E(G)$ denote vertex set and edge set of G, respectively. We denote distance between the vertices x and y in G by $d(x, y)$.

In [18], Sathyanaryana et al. introduced the concept of prime graph of an associative ring R and studied some properties. We found that a correction to be made in the statement of the Theorem 2.4 in [18]. This theorem is not true when $n = p^2$, where p is a prime. To show this we will give an example and we present the corrected theorem. We establish a formula for finding number of edges in the prime graph of $\mathbb{Z}_n$ using the gcd-sum function. We also show that the prime graph of the ring $\mathbb{Z}_n$ is same as the *order prime graph*[1] of the additive group $\mathbb{Z}_n$, when n is a prime. Further, we discuss some results regarding eigenvalues and energy of prime graphs of finite rings.

2 Preliminaries

In all our discussions, n represents a positive integer. In this section, we recall the required definitions and results.

Definition 2.1. *[18] Let R be a finite ring. The prime graph $PG(R)$ of R is defined as a graph with the vertex set $V(PG(R)) = R$ and any two vertices x and y are adjacent in $PG(R)$ if and only if $xRy = 0$ or $yRx = 0$, and $x \neq y$.*

Definition 2.2. *[17] Let Γ be a finite group. The order prime graph $OP(\Gamma)$ of Γ is defined as a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $OP(\Gamma)$ if and only if $\gcd(o(a), o(b)) = 1$ and $a \neq b$.*

Definition 2.3. *[3] The gcd-sum function is defined as the sum of the gcds of the first n positive integers with n :*

$$g(n) = \sum_{i=1}^{n} \gcd(i, n).$$

Definition 2.4. *For $n \geq 1$, let $s(n)$ denote the number of positive integers not exceeding n that satisfy the congruence $x^2 \equiv 0 \pmod{n}$.*

[1] Sattanathan and Kala have introduced the concept of order prime graphs of finite groups and studied some properties in [17].

For the first few positive integers, we have,

$$s(1) = 0, s(2) = 0, s(3) = 0, s(4) = 1, s(5) = 0,$$

$$s(6) = 0, s(7) = 0, s(8) = 1, s(9) = 2, s(10) = 0, \dots$$

We recall the following principal result on linear congruences:

Theorem 2.5. *[5, Theorem 4.7] The linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d|b$, where $d = \gcd(a, n)$. If $d|b$, then it has d mutually incongruent solutions modulo n.*

Definition 2.6. *Let G be a graph with n vertices $v_1, v_2, \dots, v_n$. The adjacency matrix $A = A(G)$ is a square matrix of order n whose (i, j)-entry is defined as*

$$A_{ij} = \begin{cases} 1, & \textit{if the vertces } v_i \textit{ and } v_j \textit{ are adjacent;} \\ 0, & \textit{otherwise.} \end{cases}$$

The eigenvalues of $A(G)$ are said to be the eigenvalues of the graph G. We denote largest and smallest eigenvalues of a graph G by λ_{max} and λ_{min} respectively.

Proposition 2.7. *[14, Proposition 2.3]*

(i) A graph is bipartite if and only if its spectrum is symmetric about the origin.

(ii) A connected graph G is bipartite if and only if $\lambda_{min}(G) = \lambda_{max}(G)$.

Definition 2.8. *[7] Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a graph G. The energy of G is defined as*

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

Let G be a graph with n vertices and m edges. The following inequalities (2) and (3) can be found in [14]:

$$max\left\{\overline{d}, \sqrt{d_{max}}\right\} \leq \lambda_{max} \leq d_{max}. \tag{2}$$

where $\overline{d}$ is the average vertex degree and d_{max} is the maximum vertex degree of G. If H is a subgraph of a graph G, then

$$\lambda_{max}(G) \geq \lambda_{max}(H). \tag{3}$$

The following inequalities are due to McClelland [15] and are obtained by Gutman [8, Theorem 5.1, Corollary 5.3.]:

$$\mathcal{E}(G) \leq \sqrt{2mn}, \tag{4}$$

$$2\sqrt{m} \leq \mathcal{E}(G) \leq 2m. \tag{5}$$

The following inequalities (6) and (7) are given by Koolen and Moulton [11, 12]:

$$\mathcal{E}(G) \leq \frac{n(\sqrt{n}+1)}{2}, \tag{6}$$

and if G is a bipartite graph with $n > 2$,

$$\mathcal{E}(G) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}}. \tag{7}$$

The following inequality gives a lower bound for the energy of G solely in terms of number of vertices [8, Theorem 5.4]:

$$\mathcal{E}(G) \geq 2\sqrt{n-1} \tag{8}$$

with equality if and only if G is the star $K_{1,n-1}$.

Definition 2.9. *Let R be a finite ring. The prime energy of the ring R, denoted by $\mathcal{PE}(R)$, is defined as the energy of the prime graph $PG(R)$. That is, $\mathcal{PE}(R) = \mathcal{E}(PG(R))$.*

3 Observation and Results

Observation 1. Let R be a finite ring. From the Definition 2.1, we have the following:

1. Since $0Rx = 0, \forall x \in R$, 0 is adjacent to every other vertex in $PG(R)$ and so $PG(R)$ is a connected graph. Also $\deg(0) = n-1$ and the maximum degree $\Delta(PG(R)) = n-1$.

2. Since 0 is adjacent to every other vertex in $PG(R)$, it follows that, $PG(R)$ contains a copy of the star $K_{1,n-1}$ and the diameter of $PG(R)$ is less than or equal to 2.

3. Since $PG(R)$ is a simple graph, we have,
$$(n-1) \leq |E(PG(R))| \leq \frac{n(n-1)}{2}.$$

4. If R is a ring with unity 1, then $PG(R)$ has atleast one pendant vertex. In particular, 1 is a pendant vertex in $PG(R)$ (See the proof of Proposition 3.13).

Observation 2. Given a ring R of finite order $n = p^2$, where p is a prime, the prime graph $PG(R)$ may or may not be a star graph. For example, consider the rings $\mathbb{Z}_4$ and $\mathbb{Z}_9$. The prime graphs $PG(\mathbb{Z}_4)$ and $PG(\mathbb{Z}_9)$ are given in Figure 1. We see that $PG(\mathbb{Z}_4)$ is a star but $PG(\mathbb{Z}_9)$ is not a star.

Proposition 3.1. *The prime graph $PG(\mathbb{Z}_{p^2})$, where p is an odd prime, has at least one triangle.*

Proof. Since p is an odd prime, $p-1 \geq 2$ and $0, p, 2p, \ldots, (p-1)p \in \mathbb{Z}_{p^2}$. Note that $(ip)\cdot(jp) \equiv 0 \pmod{p^2}$, $\forall i, j,\ 0 \leq i, j < p$. Therefore ip and jp are adjacent in $PG(\mathbb{Z}_{p^2})$ for $0 \leq i, j < p$. In particular, $0, p$ and $2p$ are mutually adjacent in $PG(\mathbb{Z}_{p^2})$. Hence $PG(\mathbb{Z}_{p^2})$ has atleast one triangle. ∎

The statement of the Theorem 2.4 in [18] is as follows: *Consider* $\mathbb{Z}_n$ *for some* n, *where* $\mathbb{Z}_n$ *is the ring of integers modulo* n. *The following conditions are equivalent:*

(i) n *is prime or* $n = p^2$ *for some prime* p

(ii) There are no triangles in $PG(\mathbb{Z}_n)$

(iii) $d(x, y) = 2$ *for any two distinct non-zero vertices of* $PG(\mathbb{Z}_n)$

(iv) $PG(\mathbb{Z}_n)$ *is a star graph with the special vertex* 0 *(the additive identity).*

By our Observation 2 and Proposition 3.1, it follows that, the theorem mentioned above is true only when n is a prime or $n = 4$. The following is the corrected version of the said theorem:

Theorem 3.2. *The following conditions are equivalent for the prime graph* $PG(\mathbb{Z}_n)$ *of* $\mathbb{Z}_n$ *:*

(i) n *is a prime or* $n = 4$.

(ii) There are no triangles in $PG(\mathbb{Z}_n)$.

(iii) $d(x, y) = 2$ *for any two distinct non-zero vertices of* $PG(\mathbb{Z}_n)$.

(iv) $PG(\mathbb{Z}_n)$ *is a star graph with the special vertex* 0 *(the zero element in* $\mathbb{Z}_n$*).*

Corollary 3.3. $PG(\mathbb{Z}_n) \cong K_{1,n-1}$ *if and only if* n *is a prime or* $n = 4$.

Proof. Follows from Theorem 3.2. □

Proposition 3.4. *Let* R_1 *and* R_2 *be two finite rings. If* $R_1 \cong R_2$ *as rings, then* $PG(R_1) \cong PG(R_2)$.

Proof. Let $f : R_1 \to R_2$ be a ring isomorphism. Clearly, f is a bijective mapping from $V(PG(R_1))$ onto $V(PG(R_2))$. Let x and y be two vertices in $PG(R_1)$. If x and y are adjacent in $PG(R_1)$, then $xR_1y = 0$ or $yR_1x = 0$. Hence $xzy = 0$ or $yzx = 0$, for all $z \in R_1$. Since f is a ring isomorphism, we have $f(x)f(z)f(y) = 0$ or $f(y)f(z)f(x) = 0$. Since f is onto it follows that $f(x)tf(y) = 0$ or $f(y)tf(x) = 0$, for all $t \in R_2$. So $f(x)R_2f(y) = 0$ or $f(y)R_2f(x) = 0$. Therefore $f(x)$ and $f(y)$ are adjacent in $PG(R_2)$. It is easy to show that, if x and y are not adjacent in $PG(R_1)$, then $f(x)$ and $f(y)$ are not adjacent in $PG(R_2)$. Thus f is a graph isomorphism of $PG(R_1)$ onto $PG(R_2)$. □

Remark 1. Converse of the Proposition 3.4 is not true. Consider the rings $\mathbb{Z}_4$ and $\mathbb{F}_4 = \mathbb{Z}_2[x]/\langle x^2 + x + 1\rangle$. Since the ring $\mathbb{Z}_4$ is of order 4, by Corollary 3.3 it follows that, $PG(\mathbb{Z}_4) \cong K_{1,3}$. Note that $\mathbb{F}_4$ is a field of order 4 whose elements can be identified by the elements $0, 1, \alpha, 1 + \alpha$ with the relations $1 + 1 = 0$ and $\alpha^2 + \alpha + 1 = 0$. Since $\mathbb{F}_4$ is a field, $1, \alpha, 1+\alpha$ are all units and so are not mutually adjacent in $PG(\mathbb{F}_4)$ (See Proposition 3.6). Since 0 is adjacent to $1, \alpha, 1 + \alpha$ in $PG(\mathbb{F}_4)$, it follows that, $PG(\mathbb{F}_4) \cong K_{1,3}$. Thus, $\mathbb{Z}_4 \cong K_{1,3} \cong PG(\mathbb{F}_4)$. But $\mathbb{Z}_4$ and $\mathbb{F}_4$ are not isomorphic as rings.

Let $(R, +, \cdot)$ be a ring. Then $(R, +)$ is an additive abelian group. By considering the additive group structure on R, we can construct the order prime graph $OP(R)$ of R. For the ring $\mathbb{Z}_n$, we have the following theorem:

Theorem 3.5. *If n is a prime, then $PG(\mathbb{Z}_n) \cong OP(\mathbb{Z}_n)$.*

Proof. Suppose that n is a prime. Then by Theorem 3.2, $PG(\mathbb{Z}_n) \cong K_{1,n-1}$ and by [17, Theorem 2.7], $OP(\mathbb{Z}_n) \cong K_{1,n-1}$. Therefore, $PG(\mathbb{Z}_n) \cong OP(\mathbb{Z}_n)$. □

Remark 2. The converse of the Theorem 3.5 is not true. If n is not a prime, then $PG(\mathbb{Z}_n)$ and $OP(\mathbb{Z}_n)$ may or may not be isomorphic. For example, consider the rings $\mathbb{Z}_6$ and $\mathbb{Z}_8$. The order prime graphs and prime graphs of the rings $\mathbb{Z}_6$ and $\mathbb{Z}_8$ are given in Figure 2 and Figure 3, respectively. We see that $PG(\mathbb{Z}_6) \cong OP(\mathbb{Z}_6)$ and $PG(\mathbb{Z}_8) \ncong OP(\mathbb{Z}_8)$.

Proposition 3.6. *The units in a ring R are not mutually adjacent in $PG(R)$.*

Proof. Let x and y be two units in R. If they are adjacent in $PG(R)$, then $xRy = 0$ or $yRx = 0$. Hence $xzy = 0$ or $yzx = 0$, for all $z \in R$. Let z be a non-zero element in R. Then

$$xzy = 0 \text{ or } yzx = 0.$$

In either case, applying left and right inverses of x and y, we obtain $z = 0$, which is a contradiction. Therefore, x and y are not adjacent in $PG(R)$. □

The following theorem gives bounds for the number of edges in the prime graph of a finite ring.

Theorem 3.7. *Let R be a finite ring of order n. We have,*

$$n - 1 \leq |E(PG(R))| \leq \binom{n}{2} - \binom{|U(R)|}{2}, \tag{9}$$

where $U(R)$ is the set of units in R.

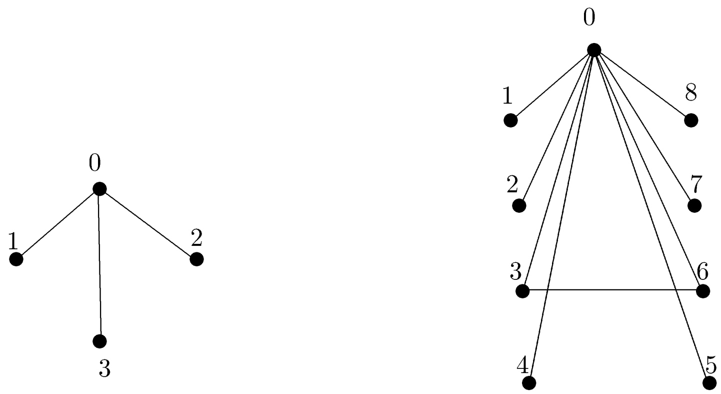
Proof. Note that $PG(R)$ is a simple graph. By Proposition 3.6, the units in R are not mutually adjacent in $PG(R)$. Therefore,

$$\begin{aligned} |E(PG(R))| &\leq \big(\text{Maximum number of edges in a simple graph with } n \text{ vertices}\big) \\ &\quad - \big(\text{Maximum number of edges in a simple graph with} \\ &\qquad |U(R)| \text{ vertices}\big) \\ &= |E(K_n)| - |E(K_{|U(R)|})| \\ &= \binom{n}{2} - \binom{|U(R)|}{2}. \end{aligned} \tag{10}$$

Since the zero element in R is adjacent to every other $(n-1)$ elements of R in the prime graph, it follows that, there are at least $(n-1)$ edges in $PG(R)$; i.e.,

$$n - 1 \leq |E(PG(R))| \tag{11}$$

Now, from the inequalities (10) and (11), (9) follows. □

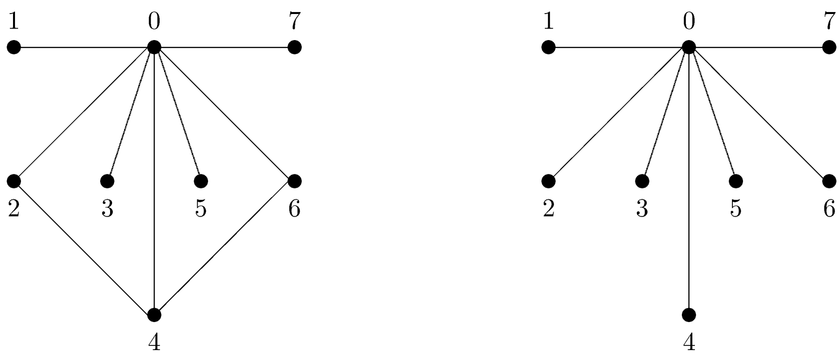


$PG(\mathbb{Z}_4)$ $PG(\mathbb{Z}_9)$

Figure 1: Prime graphs of $\mathbb{Z}_4$ and $\mathbb{Z}_9$

$PG(\mathbb{Z}_6)$ $OP(\mathbb{Z}_6)$

Figure 2: Prime graph and order prime graph of $\mathbb{Z}_6$



$PG(\mathbb{Z}_8)$ $OP(\mathbb{Z}_8)$

Figure 3: Prime graph and order prime graph of $\mathbb{Z}_8$

Corollary 3.8. *For the prime graph $PG(\mathbb{Z}_n)$,*

$$n-1 \leq |E(PG(\mathbb{Z}_n))| \leq \binom{n}{2} - \binom{\phi(n)}{2}. \tag{12}$$

Proof. Since the number of units in $\mathbb{Z}_n$ is $\phi(n)$, inequality (12) follows from the inequality (9) in Theorem 3.7. □

Notation: We denote an edge with end vertices a and b in a graph by $\overline{ab}$. The set of non-zero zero divisors in a ring R is denoted by $Z_d(R)$ and the set of units in R is denoted by $U(R)$.

Lemma 3.9. *(i) There are exactly $n-1$ edges in $PG(\mathbb{Z}_n)$ of the form $\overline{ab}$ with either a or b is zero, but not both.*

(ii) The number of edges $PG(\mathbb{Z}_n)$ of the form $\overline{ab}$ with $a \neq 0$ and $b \neq 0$ is equal to

$$\frac{1}{2}\left[\left(\sum_{j\neq 0,\ j\in Z_d(\mathbb{Z}_n)} (d_j - 1)\right) - \left(\sum_{\substack{j\neq 0,\ j\in Z_d(\mathbb{Z}_n),\\ j^2\equiv 0 \pmod n}} 1\right)\right],$$

where $d_j = \gcd(j,n)$.

Proof. (i) Since $a\cdot 0 = 0, \forall a \neq 0, a \in \mathbb{Z}_n$, it follows that 0 is adjacent to a in $PG(\mathbb{Z}_n)$, $\forall a \in \mathbb{Z}_n - \{0\}$. Since $|\mathbb{Z}_n - \{0\}| = n-1$, the result follows.

(ii) Let a be a nonzero element in $\mathbb{Z}_n$. Note that 0 is a solution of the linear congruence

$$ax \equiv 0 \pmod n. \tag{13}$$

By Theorem 2.5, there are exactly $d-1$ non-zero incongruent solutions for the linear congruence $ax \equiv 0 \pmod n$, where $d = \gcd(a,n)$. But there may be a possibility that $a\cdot a \equiv 0 \pmod n$. Note that ax in the linear congruence represents the ordered pair (a,x), but the edge $\overline{ax}$ represents the unordered pair of a and x. Also, by the definition of prime graph, $\overline{aa} \notin E(PG(\mathbb{Z}_n)), \forall a \in \mathbb{Z}_n$. Therefore, the number of edges $\overline{ab}$ with $a \neq 0$ and $b \neq 0$ is equal to

$$\frac{1}{2}\Big[\big(\text{Number of ordered pairs } (a,b) \text{ with } a \neq 0 \text{ and } b \neq 0 \text{ in } \mathbb{Z}_n \text{ such that } b \text{ is a solution of (13)}\big) - \big(\text{Number of ordered pairs } (a,a) \text{ with } a \neq 0 \text{ in } \mathbb{Z}_n \text{ such that } a\cdot a \equiv 0 \pmod n\big)\Big]$$

$$= \frac{1}{2}\left[\left(\sum_{j\neq 0,\ j\in Z_d(\mathbb{Z}_n)} (d_j - 1)\right) - \left(\sum_{\substack{j\neq 0,\ j\in Z_d(\mathbb{Z}_n),\\ j^2\equiv 0 \pmod n}} 1\right)\right],$$

where $d_j = \gcd(j,n)$. □

Theorem 3.10. *The number of edges in $PG(\mathbb{Z}_n)$ is given by*

$$|E(PG(\mathbb{Z}_n))| = \frac{1}{2}\big[g(n) - s(n) - 1\big], \tag{14}$$

where $g(n)$ and $s(n)$ are functions given in the Definitions 2.3 and 2.4.

Proof. Let $d_j = \gcd(j, n)$. Then $d_n = n$ and for $j \in U(\mathbb{Z}_n)$, $d_j = 1$. Now,

$$\begin{aligned}
|E(PG(\mathbb{Z}_n))| &= \big(\text{Number of edges in } PG(\mathbb{Z}_n) \text{ of the form } \overline{ab} \text{ with}\\
&\quad \text{either } a \text{ or } b \text{ is zero, but not both}\big) - \big(\text{Number of edges}\\
&\quad \text{in } PG(\mathbb{Z}_n) \text{ of the form } \overline{ab} \text{ with } a \neq 0 \text{ and } b \neq 0\big)\\
&= (n-1) + \frac{1}{2}\left[\left(\sum_{\substack{j\neq 0,\\ j\in Z_d(\mathbb{Z}_n)}} (d_j - 1)\right) - \left(\sum_{\substack{j\neq 0,\ j\in Z_d(\mathbb{Z}_n),\\ j^2\equiv 0 \pmod n}} 1\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{[by Lemma 3.9]}\\
&= \frac{1}{2}\left[2n - 2 + \left(\sum_{\substack{j\neq 0,\\ j\in Z_d(\mathbb{Z}_n)}} d_j\right) - \left(\sum_{\substack{j\neq 0,\\ j\in Z_d(\mathbb{Z}_n)}} 1\right) - \left(\sum_{\substack{j\neq 0,\\ j\in Z_d(\mathbb{Z}_n),\\ j^2\equiv 0 \pmod n}} 1\right)\right]\\
&= \frac{1}{2}\left[2n - 2 + \left(\sum_{\substack{j\neq 0,\\ j\in \mathbb{Z}_n}} d_j\right) - \left(\sum_{\substack{j\neq 0,\\ j\in U(\mathbb{Z}_n)}} d_j\right) - |Z_d(\mathbb{Z}_n)| - s(n)\right]\\
&= \frac{1}{2}\left[2n - 2 + \left(\sum_{0<j<n} d_j\right) - |U(\mathbb{Z}_n)| - |Z_d(\mathbb{Z}_n)| - s(n)\right]\\
&= \frac{1}{2}\left[2n - 2 + \left(\sum_{j=0}^{n} d_j\right) - d_n - \Big(|U(\mathbb{Z}_n)| + |Z_d(\mathbb{Z}_n)|\Big) - s(n)\right]\\
&= \frac{1}{2}\left[2n - 2 + g(n) - n - (n-1) - s(n)\right]\\
&\qquad\qquad \big(\because |U(\mathbb{Z}_n)| + |Z_d(\mathbb{Z}_n)| = |\mathbb{Z}_n - \{0\}| = n - 1\big)\\
&= \frac{1}{2}\big[g(n) - s(n) - 1\big].
\end{aligned}$$

□

Theorem 3.11. *If R is a finite ring of order n, then*

$$max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} \leq \lambda_{max}(PG(R)) \leq n - 1. \tag{15}$$

In particular, if $n \geq 3$, $\sqrt{n-1} \leq \lambda_{max}(PG(R)) \leq n - 1$.

Proof. Since $d_{max}(PG(R)) = n-1$, $K_{1,n-1}$ is a subgraph of $PG(R)$. Therefore from the inequalities (2) and (3), it follows that,

$$max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} \leq \lambda_{max}(K_{1,n-1})) \leq \lambda_{max}(PG(R)) \leq n-1.$$

Clearly, for $n \geq 3$, $max\left\{\frac{2n-2}{n}, \sqrt{n-1}\right\} = \sqrt{n-1}$, and from (15), we have, $\sqrt{n-1} \leq \lambda_{max}(PG(R)) \leq n-1$. □

Theorem 3.12. *Let R be a finite ring. Then*

(i) $|R|$ is a prime or $|R| = 4$ if and only if for each eigenvalue λ of $PG(R)$, $-\lambda$ is an eigenvalue with the same multiplicity.

(ii) $|R|$ is a prime or $|R| = 4$ if and only if $\lambda_{min}(PG(R)) = -\lambda_{max}(PG(R))$.

Proof. By Corollary 3.3, $|R|$ is a prime or $|R| = 4$ if and only if $PG(R) \cong K_{1,n-1}$, a bipartite graph. Hence by Proposition 2.7, the proof of (i) and (ii) follows. □

Proposition 3.13. *If R is a finite ring with unity 1, then the diameter of $PG(R)$ is 2.*

Proof. Note that 0 is adjacent to every vertex x ($x \neq 0$) in $PG(R)$. Therefore $d(x, y) \leq 2, \forall x, y \in R$. Also, for all $x \neq 0$ in R, $xR1 \neq 0$ and $0R1 = 0$. Hence 1 is adjacent only to 0 in $PG(R)$. It follows that, 1 is a pendent vertex in $PG(R)$ and $d(1, x) = 2, \forall x \neq 0, 1$. Therefore the diameter of $PG(R)$ is 2. □

Theorem 3.14. *If R is a finite ring with unity 1 and $|R| \geq 3$, then $PG(R)$ has atleast three distinct eignvalues.*

Proof. A connected graph with diameter d, has at least $d+1$ distinct eigenvalues [4, Proposition 1.3.3, p.5]. Since $|R| \geq 3$, by Proposition 3.13, it follows that, the diameter of $PG(R)$ is 2. Hence $PG(R)$ has atleast three distinct eignvalues. □

Theorem 3.15. *Let R be a finite ring.*

(i) If $|R| = 2$, then $PG(R) \cong K_2$ and $\mathcal{PE}(R) = 2$.

(ii) If $|R| = n$, then

$$\mathcal{PE}(R) \leq \frac{n(\sqrt{n}+1)}{2} \tag{16}$$

and

$$\mathcal{PE}(R) \geq 2\sqrt{n-1}. \tag{17}$$

(iii) If n is a prime or $n = 4$, then

$$\mathcal{PE}(\mathbb{Z}_n) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \tag{18}$$

and

$$\mathcal{PE}(\mathbb{Z}_n) = 2\sqrt{n-1}. \tag{19}$$

(iv) For any positive integer n,

$$\mathcal{PE}(\mathbb{Z}_n) \leq \sqrt{\frac{n}{2}\big[g(n) - s(n) - 1\big]} \tag{20}$$

and

$$2\sqrt{\frac{1}{2}\big[g(n) - s(n) - 1\big]} \leq \mathcal{PE}(\mathbb{Z}_n) \leq \big[g(n) - s(n) - 1\big]. \tag{21}$$

Proof.

(i) Follows by direct computation.

(ii) Follows from the Koolen and Moulton inequality (6).

(iii) If n is a prime or $n = 4$, then by Corollary 3.3, $PG(\mathbb{Z}_n) \cong K_{1,n-1}$, a bipartite graph. Now, the inequality (18) follows from the Koolen and Moulton inequality (7), and eq.(19) follows from (8) with equality.

(iv) Let m be the number of edges in $PG(\mathbb{Z}_n)$). Then by Theorem 3.10,

$$m = \frac{1}{2}\big[g(n) - s(n) - 1\big]$$

Using this in inequalities (4) and (5), we obtain the inequality (20) and (21), respectively.

□

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